Pergamon

Computers Math. Applic. Vol. 28, No. 1–3, pp. 71–79, 1994

0898-1221/94 $7.00 + 0.00

0898-1221(94)00094-8

Monotone Solutions of a Class of Nonlinear Difference Equations

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Abstract—Comparison theorems and existence criteria are summarized and new ones are derived for the monotone solutions of a class of nonlinear difference equations which arises from Hardy's inequality.

Keywords—Nonlinear difference equations, Comparison theorems, Existence criteria, Banach contraction principle.

SECTION 1.

Sufficient as well as necessary conditions for the existence of monotone solutions have been derived for various classes of differential and difference equations (see, for example [1–7]). In this paper, we are also concerned with the existence conditions for a class of nonlinear difference equations of the form

$$\Delta\left(p_{k-1}\left(\Delta x_{k-1}\right)^{\sigma}\right)+q_k\, x_k^{\sigma}=0, \qquad k=1,2,3,\dots, \tag{1}$$

where $\sigma > 0$ and $\{p_k\}_0^{\infty}$ is a positive sequence. In [8], it is shown that Hardy's inequality for a series [9, pp. 239–241] can be viewed as a necessary condition for the existence of a positive nondecreasing solution of (1). In the case of $\sigma = 1$ and $q_i \geq 0$, Equation (1) changes to a linear second order difference equation,

$$\Delta\left(p_k\, \Delta x_{k-1}\right)+q_k\, x_k=0, \qquad k=1,2,3,\dots,$$

with respect to which the existence of a positive nondecreasing solution is equivalent to the existence of a positive solution (see, for example, [10, Lemma 1]).

By means of two Riccati-type transformations, we shall first establish several equivalent conditions for the existence of monotone solutions of (1). Then we establish some comparison theorems. These theorems state roughly this: if there is a positive nondecreasing solution for a "majorant equation" of the form

$$\Delta\left(R_{k-1}\left(\Delta y_{k-1}\right)^{\sigma}\right)+\lambda_k\, S_k\, y_k^{\sigma}=0, \qquad k=1,2,3,\dots, \tag{2}$$

then a minorant equation,

$$\Delta\left(r_{k-1}\left(\Delta z_{k-1}\right)^{\sigma}\right)+s_k\, z_k^{\sigma}=0, \qquad k=1,2,3,\dots, \tag{3}$$

The second author is partially supported by NNSF of China.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX also has a positive nondecreasing solution. The precise conditions on the coefficients of the involved equations will be stated later.

The successful use of comparison theorems as means to derive existence criteria for positive nondecreasing solutions depends very much on specific equations, which are known to possess (or not to possess) monotone solutions. In Section 3, we shall mention a few existence criteria which are known [1,2,10,11], and derive an additional criterion by means of the Banach contraction principle.

SECTION 2.

As mentioned before, σ will denote a positive number. We begin by assuming $\{x_k\}_0^\infty$ is a positive nondecreasing solution of (1). Then the sequence $\{u_k\}_0^\infty$ defined by

$$u_k = \frac{(\Delta x_k)^\sigma}{x_k^\sigma}, \qquad k \geq 0,$$

is nonnegative and we may easily verify that it satisfies

$$u_{k+1} = \frac{p_k}{p_{k+1}} f(u_k) - \frac{q_{k+1}}{p_{k+1}}, \qquad k \geq 0, \tag{4}$$

where

$$f(u) = \frac{u}{\left(1 + u^{1/\sigma}\right)^\sigma}. \tag{5}$$

Furthermore, the sequence $\{v_k\}_0^\infty$ defined by

$$v_k = \frac{p_k\,(\Delta x_k)^\sigma}{x_k^\sigma}, \qquad k \geq 0,$$

is nonnegative and satisfies

$$\Delta v_k + F(p_k, v_k) + q_{k+1} = 0, \qquad k \geq 0, \tag{6}$$

where

$$F(t,v) = \frac{v\left[\left(t^{1/\sigma} + v^{1/\sigma}\right)^\sigma - t\right]}{\left(t^{1/\sigma} + v^{1/\sigma}\right)^\sigma}. \tag{7}$$

Conversely, if $\{u_k\}_0^\infty$ is a nonnegative solution of (4), then defining $x_0 = 1$ and

$$x_{k+1} = \left(1 + u_k^{1/\sigma}\right) x_k, \qquad k > 0,$$

we may easily verify that $\{x_k\}_0^\infty$ is a positive nondecreasing solution of (1). Similarly, if $\{v_k\}_0^\infty$ is a nonnegative solution of (6), then defining $x_0 = 1$, and

$$x_{k+1} = \left[1 + \left(\frac{v_k}{p_k}\right)^{1/\sigma}\right] x_k, \qquad k \geq 0,$$

we may easily verify that $\{x_k\}_0^\infty$ is a positive nondecreasing solution of (1).

LEMMA 1. *Equation (1) has a positive nondecreasing solution if and only if (4) (or (6)) has a nonnegative solution.*

Similar reasoning readily leads to the following lemma.

LEMMA 2. *Suppose $\sigma > 0$ and $\{p_k\}_0^\infty$ is a positive sequence. The recurrence relation*

$$\Delta\left(p_{k-1}\,(\Delta x_{k-1})^\sigma\right) + q_k\, x_k^\sigma \leq 0, \qquad k = 1, 2, 3, \ldots, \tag{8}$$

has a positive nondecreasing solution if and only if

$$u_{k+1} \le \frac{p_k}{p_{k+1}} f(u_k) - \frac{q_{k+1}}{p_{k+1}}, \qquad k \ge 0, \qquad or \tag{9}$$

$$\Delta v_{k+1} + F(p_k, v_k) + q_{k+1} \le 0, \qquad k \ge 0, \tag{10}$$

has a nonnegative solution.

The functions f and F defined, respectively, by (5) and (7) have several monotonicity and homogeneity properties which we shall need later.

LEMMA 3. *The functions f and F satisfy*

(i) $$f'(u) = \left(1 + u^{1/\sigma}\right)^{-\sigma-1},$$

(ii) $$F_t(t,v) = \frac{-v^{1+1/\sigma}}{\left(t^{1/\sigma} + v^{1/\sigma}\right)^{1+\sigma}},$$

(iii) $$F_v(t,v) = \frac{\left(t^{1/\sigma} + v^{1/\sigma}\right)^{1+\sigma} - \left(t^{1/\sigma}\right)^{1+\sigma}}{\left(t^{1/\sigma} + v^{1/\sigma}\right)^{1+\sigma}}, \qquad and$$

(iv) $$\alpha F(t,v) = F(\alpha t, \alpha v),$$

where the formulae are valid whenever the quantities involved are defined.

The recurrence relation (8) and the difference equation (1) are related in the following manner.

THEOREM 1. *If the recurrence relation (8) has a positive nondecreasing solution, so does Equation (1). The converse is obviously true.*

PROOF. Suppose the recurrence relation (8) has a positive nondecreasing solution, then by Lemma 2, the recurrence relation (9) has a nonnegative solution $\{u_k\}_0^\infty$. By Lemma 1, it suffices to show that

$$w_{k+1} = \frac{p_k}{p_{k+1}} f(w_k) - \frac{q_{k+1}}{p_{k+1}}, \qquad k \ge 0, \tag{11}$$

has a solution $\{w_k\}_0^\infty$ such that $w_k \ge u_k$, for $k \ge 0$. To construct such a solution, we choose $w_0 \ge u_0$, and define w_k by (11), so that from Lemma 3, it follows that

$$w_1 - u_1 = \frac{p_0}{p_1} \left(f(w_0) - f(u_0)\right) = \frac{p_0}{p_1} f'(\tau)\,(w_0 - u_0) \ge 0,$$

where $0 \le u_0 \le \tau$. An easy induction then shows that $w_n - u_n \ge 0$ for $n > 1$. ∎

THEOREM 2. *(cf. [2, Theorem 1]). Suppose $\sigma > 0, \{r_k\}_0^\infty$ and $\{R_k\}_0^\infty$ are positive sequences such that $0 < R_k \le r_k$ for $k \ge 0$. Suppose further that $s_k \le S_k$, $\lambda_k \ge 1$ and $\Delta\lambda_k \ge 0$, for $k \ge 1$. If Equation (2) has a positive nondecreasing solution, so does (3).*

PROOF. By our assumptions, the equation

$$\Delta v_k + F(R_k, v_k) + \lambda_{k+1} S_{k+1} = 0, \qquad k \ge 0, \tag{12}$$

has a nonnegative solution $\{v_k\}_0^\infty$. Thus, dividing Equation (12) by λ_{k+1}, we obtain, in view of Lemma 3, that

$$\frac{\Delta v_k}{\lambda_{k+1}} + F\left\{\frac{R_k}{\lambda_{k+1}}, \frac{v_k}{\lambda_{k+1}}\right\} + S_{k+1} = 0, \qquad k \ge 0.$$

But since $S_k \ge s_k$,

$$\frac{\Delta v_k}{\lambda_{k+1}} \ge \Delta\left\{\frac{v_k}{\lambda_k}\right\},$$

and (by Lemma 3)

$$F\left\{\frac{R_k}{\lambda_{k+1}}, \frac{v_k}{\lambda_k+1}\right\} \geq F\left\{\frac{r_k}{\lambda_{k+1}}, \frac{v_k}{\lambda_{k+1}}\right\} \geq F\left\{r_k, \frac{v_k}{\lambda_k}\right\},$$

thus

$$\Delta\left\{\frac{v_k}{\lambda_k}\right\} + F\left\{r_k, \frac{v_k}{\lambda_k}\right\} + s_{k+1} \leq 0, \qquad k \geq 0,$$

which implies that

$$\Delta z_k + F(r_k, z_k) + s_{k+1} \leq 0, \qquad k \geq 0$$

has a nonnegative solution. By Lemmas 1 and 2, Equation (3) has a positive nondecreasing solution. ∎

There is a dual to the above theorem as follows.

THEOREM 3. *Suppose $\sigma > 0$, $\{r_k\}_0^\infty$ and $\{R_k\}_0^\infty$ are positive sequences such that $0 < r_k \leq R_k$ for $k \geq 0$. Suppose further that $s_k \geq S_k$ and $0 < \lambda_k \leq 1$, $\Delta\lambda_k \leq 0$, for $k \geq 1$. If Equation (3) has a positive nondecreasing solution, so does (2).*

PROOF. By our assumptions, the equation

$$\Delta v_k + F(r_k, v_k) + s_{k+1} = 0, \qquad k \geq 0,$$

has a nonnegative solution $\{v_k\}_0^\infty$. It thus follows that

$$\Delta v_k + F(r_k, v_k) + S_{k+1} \leq 0, \qquad k \geq 0,$$

which, after multiplying by λ_{k+1}, becomes

$$\lambda_{k+1}\,\Delta v_k + F\left(\lambda_{k+1}\, r_k,\ \lambda_{k+1}\, v_k\right) + \lambda_{k+1}\, S_{k+1} \leq 0, \qquad k \geq 0.$$

Since

$$\lambda_{k+1}\Delta v_{k+1} \geq \Delta\left(\lambda_k\, v_k\right)$$

and (by Lemma 3)

$$F\left(\lambda_{k+1}\, r_k,\ \lambda_{k+1}\, v_k\right) \geq F\left(r_k,\ \lambda_{k+1}\, v_k\right) \geq F\left(R_k,\ \lambda_{k+1}\, v_k\right) \geq F\left(R_k,\ \lambda_k\, v_k\right),$$

thus

$$\Delta\left(\lambda_k\, v_k\right) + F\left(R_k, \lambda_k\, v_k\right) + \lambda_{k+1}\, S_{k+1} \leq 0, \qquad k \geq 0.$$

This implies that

$$\Delta w_k + F(R_k, w_k) + \lambda_{k+1}\, S_{k+1} \leq 0, \qquad k \geq 0,$$

has a nonnegative solution, and hence Equation (2) has a positive nondecreasing solution. ∎

SECTION 3.

Existence and nonexistence of positive nondecreasing solutions of particular cases of the difference equation of the form (1) are known. For the sake of completeness, we summarize them as Theorems 4–6 as follows.

THEOREM 4. *(Cheng and Patula [10]). Suppose $s_k \geq 0$ for $k \geq 1$ and*

$$\sum_{k=n}^{\infty} s_{k+1} \leq \frac{1}{e^{2\sigma}\, 2^{n\sigma}}, \qquad n = 0, 1, 2, \ldots,$$

then the equation

$$\Delta\left(\Delta x_{k-1}\right)^{\sigma} + s_k\, x_k^{\sigma} = 0, \qquad k \geq 1; \qquad \sigma > 0, \tag{13}$$

has a positive nondecreasing solution.

THEOREM 5. *(Cheng and Lu [8]). The sequence* $\left\{k^{\sigma/(1+\sigma)}\right\}_0^{\infty}$ *is a positive nondecreasing solution of the recurrence relation*

$$\Delta\left(\Delta x_{k-1}\right)^{\sigma} + \left\{\frac{\sigma}{1+\sigma}\right\}^{1+\sigma}\left\{\frac{1}{k}\right\}^{1+\sigma} x_k^{\sigma} < 0, \qquad k = 1, 2, 3, \dots.$$

As a consequence, the equation

$$\Delta\left(\Delta x_{k-1}\right)^{\sigma} + \left\{\frac{\sigma}{1+\sigma}\right\}^{1+\sigma}\left\{\frac{1}{k}\right\}^{1+\sigma} x_k^{\sigma} = 0, \qquad k = 1, 2, 3, \dots$$

has a positive nondecreasing solution by means of Theorem 1.

THEOREM 6. *(Cheng and Zhang [11]). Equation (13) cannot have a positive nondecreasing solution when* $s_k \geq 0$, *for* $k \geq 1$, *and*

$$\sum_{k=n}^{\infty} s_{k+1} \geq \frac{\alpha^{\sigma}}{(n+1)^{\sigma}}, \qquad \text{where } \alpha > \frac{\sigma}{(1+\sigma)^{1+1/\sigma}}.$$

Theorem 6 is based on a nonexistence criterion [11] which is meant for difference equations of the form (13). This criterion can also be extended to suit the more general Equation (1) when $\{q_n\}$ is a nonnegative sequence. Let

$$\phi_n^0 = \sum_{i=n}^{\infty} q_{i+1}, \qquad n \geq 0, \qquad \text{and} \tag{14}$$

$$\phi_n^{j+1} = \phi_n^0 + \sum_{i=n}^{\infty} F\left(p_i, \phi_i^j\right), \qquad n \geq 0; \quad j \geq 0. \tag{15}$$

THEOREM 7. *Suppose* $\{q_n\}$ *is a nonnegative sequence. If Equation (1) has a positive nondecreasing solution, then the double sequence* $\left\{\phi_n^j\right\}$ *defined by (14) and (15) satisfies (i)* $\phi_n^j < \infty$ *for every* $j \geq 0$ *and* $n \geq 0$; *and (ii)* $\limsup_{j\to\infty} \phi_n^j < \infty$, *for all* $n \geq 0$.

The proof of this result is similar to that of Cheng and Zhang [11, Theorem 3] and therefore omitted.

COROLLARY 1. *Suppose* $\{q_n\}_1^{\infty}$ *is a nonnegative sequence and*

$$\sum_{i=1}^{\infty} q_i = \infty, \qquad \text{or}$$

$$\sum_{i=0}^{\infty} F\left(p_i, \sum_{j=i}^{\infty} q_{j+1}\right) = \infty,$$

then Equation (1) has no positive nondecreasing solutions.

THEOREM 8. *Suppose* $\{q_n\}$ *is a nonnegative sequence,*

$$\sum_{i=0}^{\infty} p_i^{-1/\sigma} < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} q_i\, \Gamma_i^{\tau} = \infty,$$

where

$$\Gamma_n = \sum_{i=n}^{\infty} p_i^{-1/\sigma}, \quad \tau = \sigma, \quad \textit{if } \sigma \geq 1, \qquad \textit{and } \tau = 1, \quad \textit{if } 0 < \sigma \leq 1;$$

then Equation (1) cannot have a positive nondecreasing solution.

PROOF. Suppose, to the contrary, that $\{x_n\}$ is a positive nondecreasing solution of (1). Then there is a number μ in (0,1) such that $x_n \geq \mu$, for all large n. Also, by our assumptions on $\{p_n\}$, $\Gamma_n \leq \mu$, for all large n. Consequently, if $\sigma \geq 1$, we have $x_n^\sigma \geq \mu^\sigma \geq \Gamma_n^\sigma$, for all large n; and if $0 < \sigma < 1$, we have $x_n^\sigma \geq \mu^\sigma \geq \mu \geq \Gamma_n$, for all large n. Next, from (1), we obtain

$$p_{n+1} (\Delta x_{n+1})^\sigma - p_N (\Delta x_N)^\sigma + \sum_{i=N}^{n} q_{i+1} x_{i+1}^\sigma = 0,$$

where $n \geq N \geq 0$. Hence, for sufficiently large N,

$$p_N (\Delta x_N)^\sigma \geq \sum_{i=N}^{n} q_{i+1} x_{i+1}^\sigma \geq \sum_{i=N}^{n} q_{i+1} \Gamma_{i+1}^\tau \longrightarrow \infty,$$

as $n \to \infty$, which is a contradiction. ∎

In the remainder of this paper, we shall derive an existence criterion based on the Banach contraction principle. For this purpose, we shall need the following lemma, the proof of which can be found in [9, Theorem 41].

LEMMA 4. *Suppose $x, y \geq 0$. If $\tau > 1$, then $x^\tau - y^\tau \leq \tau x^{\tau-1} (x - y)$; and if $0 < \tau \leq 1$, then $x^\tau - y^\tau \leq \tau y^{\tau-1} (x - y)$.*

For convenience, let us denote $F(1, v)$ by $G(v)$, where $F(t, v)$ has been defined by (7). Note that if $v \geq 0$, then by Lemma 4, if $\sigma > 1$, we have

$$\left(1 + v^{1/\sigma}\right)^\sigma - 1 \leq \sigma \left(1 + v^{1/\sigma}\right)^{\sigma-1} v^{1/\sigma}, \qquad \text{so that}$$

$$G(v) = \frac{v\left[\left(1 + v^{1/\sigma}\right)^\sigma - 1\right]}{\left(1 + v^{1/\sigma}\right)^\sigma} \leq \frac{\sigma v^{1+1/\sigma}}{\left(1 + v^{1/\sigma}\right)};$$

and if $0 < \sigma \leq 1$, we have

$$\left(1 + v^{1/\sigma}\right)^\sigma - 1 \leq \sigma v^{1/\sigma}, \qquad \text{so that}$$

$$G(v) \leq \frac{\sigma v^{1+1/\sigma}}{\left(1 + v^{1/\sigma}\right)^\sigma}.$$

To summarize, if $v \geq 0$, then

$$G(v) \leq \sigma v^{1+1/\sigma}. \tag{16}$$

Similarly, if $v \geq 0$, then

$$\left(1 + v^{1/\sigma}\right)^{1+\sigma} - 1 \leq (1 + \sigma) \left(1 + v^{1/\sigma}\right)^\sigma v^{1/\sigma}, \qquad \text{so that}$$

$$0 \leq G'(v) = \frac{\left(1 + v^{1/\sigma}\right)^{1+\sigma} - 1}{\left(1 + v^{1/\sigma}\right)^{1+\sigma}} \leq \frac{(1 + \sigma) v^{1/\sigma}}{\left(1 + v^{1/\sigma}\right)}. \tag{17}$$

We remark in passing that $G'(v)$ is nondecreasing for $v \geq 0$. This fact will be used later.

LEMMA 5. *Let $\{s_k\}_1^\infty$ be a nonnegative sequence such that $s_k \neq 0$ for infinitely many k. Let*

$$\rho_k = \sum_{k=n}^{\infty} s_{k+1}, \qquad n \geq 0,$$

and suppose

$$\begin{aligned} &\sum_{k=0}^{\infty} \rho_k^{1/\sigma} < \frac{1}{(1+\sigma)2^{1/\sigma}}, \quad \text{if } 0 < \sigma \leq 1, \qquad \text{and} \\ &\sum_{k=0}^{\infty} \rho_k^{1/\sigma} \leq \frac{1}{(2\sigma)2^{1/\sigma}}, \qquad \text{if } \sigma > 1, \end{aligned} \tag{18}$$

then the equation

$$w_n = \rho_n + \sum_{k=n}^{\infty} G(w_k), \qquad n = 0, 1, 2, \dots, \tag{19}$$

has a positive solution $\{w_k\}_0^\infty$ which converges to zero.

PROOF. Let B be the subspace of the vector space l^∞ consisting of real sequences $y = \{y_k\}_0^\infty$ such that $\{|y_k|/\rho_k\}_0^\infty$ is bounded. Then it is easily verified that B equipped with the norm

$$||y|| = \sup_{k\geq 0} \left(\frac{|y_k|}{\rho_k}\right)$$

is a Banach space. We define a subset Ω of B as follows:

$$\Omega = \left\{ y \in B \mid 1 \leq \frac{y_k}{\rho_k} \leq 2, \quad \text{for } k \geq 0 \right\}.$$

It is easily seen that Ω is a closed and bounded subset of B. Define the mapping T from Ω into B as follows: for any $w \in \Omega$,

$$(T\,w)_n = \rho_n + \sum_{k=n}^{\infty} G(w_k), \qquad n \geq 0.$$

For any $w \in \Omega$, since $w_k \geq \rho_k > 0$, for $k \geq 0$, we have $G(w_k) > 0$ for $k \geq 0$ so that $(T\,w)_n \geq \rho_n$ for $n \geq 0$. Furthermore, since $w_k \leq 2\rho_k$ for $k \geq 0$, we have, in view of (16), that

$$G(w_k) \leq G(2\rho_k) < \sigma\,(2\rho_k)^{1+1/\sigma}.$$

Thus by (18),

$$(T\,w)_n \leq \rho_n + \sum_{k=n}^{\infty} G(2\rho_k) < \rho_n + (2\sigma)\,2^{1/\sigma} \sum_{k=n}^{\infty} \rho_k^{1+1/\sigma} \leq \rho_n + (2\sigma)\,2^{1/\sigma}\,\rho_n \sum_{k=n}^{\infty} \rho_k^{1/\sigma} \leq 2\rho_k,$$

for $n \geq 0$.

We have thus shown that T maps Ω into Ω. Next, we shall show that T is a contraction mapping on Ω. To see this, let y and z be any two elements in Ω. By means of (17), we see from the monotonicity of $G'(v)$ and (17) that

$$\begin{aligned} |(T\,y)_n - (T\,z)_n| \leq \sum_{k=n}^{\infty} |(G\,y)_k - (G\,z)_k| &\leq \sum_{k=n}^{\infty} |G'(2\rho_k)|\,|y_k - z_k| \\ &\leq (1+\sigma)\,2^{1/\sigma} \sum_{k=n}^{\infty} \rho_k^{1/\sigma}\,|y_k - z_k|, \end{aligned}$$

for $n \geq 0$. Thus by (18),

$$\frac{|(T\,y)_n - (T\,z)_n|}{\rho_n} \leq (1+\sigma)\,2^{1/\sigma}\|y-z\| \sum_{k=n}^{\infty} \rho_k^{1/\sigma} \leq \lambda\,\|y-z\|,$$

where $\lambda < 1$, as required.

By Banach's contraction principle, there is a unique fixed point $w = \{w_k\}_0^\infty$ in Ω such that $T\,w = w$. This implies that w satisfies (19) and $\rho_n \leq w_n \leq 2\rho_n$, for $n \geq 0$, so that $\{w_n\}$ converges to zero as required. ∎

Assuming the existence of a fixed point $\{w_n\}_0^\infty$ for the operator T, if we define

$$x_0 = 1 \quad \text{and} \quad x_{n+1} = x_n\left(1 + w_n^{1/\sigma}\right), \qquad n \geq 0, \tag{20}$$

then it is easily verified that $\{x_n\}_0^\infty$ is a positive increasing solution of (13).

THEOREM 9. *Suppose the conditions of Lemma 5 hold, then Equation (13) has a positive increasing solution $\{x_k\}_0^\infty$ which converges to a finite limit.*

It remains to show that the sequence $\{x_k\}_0^\infty$ defined by (20) has a finite limit. In fact, since

$$\sum_{i=0}^{n} w_i^{1/\sigma} \leq 2^{1/\sigma} \sum_{i=0}^{n} \rho_i^{1/\sigma} < \infty,$$

we see from

$$x_{n+1} = \prod_{i=o}^{n}\left(1 + w_i^{1/\sigma}\right), \qquad n \geq 0,$$

that $\{x_k\}_0^\infty$ converges as desired.

It is interesting to compare the two existence Theorems 4 and 9. First of all, Theorem 4 is obtained by means of the Schauder fixed point theorem, while we use Banach's contraction principle to derive Theorem 9. Second, neither Theorem 4 implies Theorem 9, nor Theorem 9 implies Theorem 4. Indeed, the condition in Theorem 4 is equivalent to

$$\rho_k^{1/\sigma} \leq \frac{1}{2^k\,e^2}, \qquad k \geq 0,$$

which implies

$$\sum_{k=0}^{\infty} \rho_k^{1/\sigma} \leq \frac{2}{e^2} \cong 0.2707.$$

However, the maximum of the function

$$h(\sigma) = \begin{cases} \frac{1}{(1+\sigma)\,2^{1/\sigma}}, & 0 < \sigma \leq 1, \\ \frac{1}{(2\sigma)\,2^{1/\sigma}}, & \sigma > 1, \end{cases}$$

is equal to 0.25.

It will be of interest to establish existence theorems which include both of these results.

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